VIII. Indefinite integrals and the substitution rule

We denote the set of all antiderivatives of $f(x)$ by $\int f(x)\,dx$. We call this set the indefinite integral of $f(x)$. Notice that it suffices to find just one antiderivative $F(x)$ with $F'(x) = f(x)$ because then all the other antiderivatives of $f(x)$ are obtained by adding a constant to $F(x)$. In other words,

$$\int f(x)\,dx = F(x) + C.$$

Example Find the following indefinite integrals

(a) $\displaystyle\int 3x^4 + 5x^2 + 3\,dx$

(b) $\displaystyle\int 5\sqrt{x} - \frac{2}{\sqrt[3]{x}}\,dx.$

Solution.

The substitution rule for indefinite integrals

Theorem 1 If f is a continuous function and $u = g(x)$ is a differentiable function then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof Suppose $F(x)$ is an antiderivative of $f(x)$. Then by the chain rule

$$\frac{d}{dx}\Big[F(g(x))\Big] = F'(g(x))g'(x) = f(g(x))g'(x).$$

Hence,

$$\begin{aligned}\int f(g(x))g'(x)\,dx &= \int \frac{d}{dx}\Big[F(g(x))\Big]\,dx \\ &= F(g(x)) + C = F(u) + C \\ &= \int F'(u)\,du = \int f(u)\,du.\end{aligned}$$

□

Example Find $\int \sec^2(5t+1)\,dt$.

Solution:

Example Evaluate $\int x\sqrt{-3x+2}\,dx$.

Solution:

Example Find $\int \sin^2 x \, dx$.

Solution:

The substitution rule for definite integrals

Theorem 2 If g' exists and is continuous on $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))g'(x)\, dx = \int_{u=g(a)}^{g(b)} f(u)\, du.$$

PROOF Let $F(x)$ be an antiderivative of $f(x)$. Then by the chain rule $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ and

$$\int_a^b f(g(x))g'(x)\, dx = \Big[F(g(x))\Big]_{x=a}^{x=b} = F(g(b)) - F(g(a))$$

$$= \Big[F(u)\Big]_{u=g(a)}^{u=g(b)} = \int_{u=g(a)}^{g(b)} f(u)\, du.$$

□

Example Compute $\int_{-1}^{1} 3x^2 \sqrt{x^3 + 1}\, dx$.

Solution:

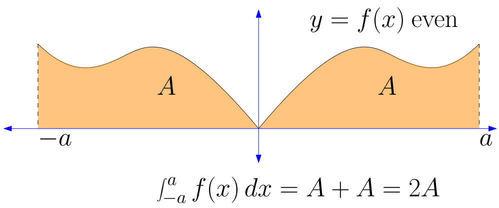
Definite integrals of symmetric functions

Theorem 3 Let $f(x)$ be a continuous function on $[-a, a]$.

(a) If f is even then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

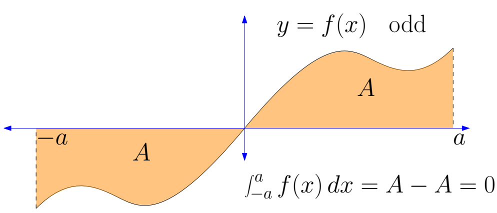
(b) If f is odd then
$$\int_{-a}^{a} f(x)\,dx = 0.$$

PROOF **Proof:** a)



$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

Proof of b) is similar.



□

8. Tutorial exercises

Work in groups:

Exercise 1. Find the general indefinite integrals.

a) $\displaystyle\int (5 + 2\sqrt{x})\, dx.$

b) $\displaystyle\int (x + \cos x)\, dx.$

c) $\displaystyle\int \sqrt{t}(t^2 + 3t + 2)\, dt.$

Exercise 2. Evaluate the definite integrals.

a) $\displaystyle\int_0^8 \frac{1}{8} + \frac{1}{2}w + \frac{1}{3}w^{1/3}\, dw.$

b) $\displaystyle\int_1^3 \frac{3x^2 + 4x + 1}{x}\, dx.$

c) $\displaystyle\int_0^{\pi/2} (\sqrt{t} - 3\cos t)\, dt.$

d) $\displaystyle\int_0^{\pi/4} 3e^x - 4\sec x \tan x\, dx.$

Exercise 3. Evaluate the integrals by making the given substitution.

a) $\displaystyle\int xe^{-x^2}\, dx, \quad u = -x^2.$

b) $\displaystyle\int \sin^2\theta \cos\theta\, d\theta, \quad u = \sin\theta.$

c) $\displaystyle\int \frac{x^3}{x^4 - 5}\, dx, \quad u = x^4 - 5.$

Exercise 4. Evaluate the definite integrals.

a) $\displaystyle\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\, dt.$

b) $\displaystyle\int_1^4 \frac{\sqrt{2 + \sqrt{x}}}{\sqrt{x}}\, dx.$

c) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\, dx.$

d) $\displaystyle\int_0^a x\sqrt{x^2 + a^2}\, dx, \quad a > 0.$

Take-home problems:

Exercise 5. Find the general indefinite integrals.

a) $\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2+1}\right)\,dx.$

b) $\displaystyle\int (2 + 3^x)\,dx.$

c) $\displaystyle\int (2 + \tan^2\theta)\,d\theta.$

Exercise 6. Evaluate the definite integrals.

a) $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta.$

b) $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta.$

c) $\displaystyle\int_{-1}^{2} x - 2|x|.$

d) $\displaystyle\int_0^{3\pi/2} |\sin(x)|\,dx.$

Exercise 7. Evaluate the integrals by making the given substitution.

a) $\displaystyle\int \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx, \quad u = 1 + \frac{1}{x}.$

b) $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt, \quad u = \sqrt{t}.$

Exercise 8. Evaluate the definite integrals.

a) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz.$

b) $\displaystyle\int_1^4 \frac{1}{(x+1)\sqrt{x}}\,dx.$

c) $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}.$

d) $\displaystyle\int_1^{16} \frac{x^{1/2}}{1+x^{3/4}}\,dx.$

Answers

1
a) $5x + \frac{4}{3}x\sqrt{x} + C$
b) $\frac{1}{2}x^2 + \sin(x) + C$
c) $\frac{2}{7}t^{7/2} + \frac{6}{5}t^{5/2} + \frac{4}{3}t^{3/2} + C$

2
a) 21
b) $20 + \ln(3)$
c) $\frac{\pi}{3}\sqrt{\frac{\pi}{2}} - 3$
d) $3e^{\pi/4} + 1 - 4\sqrt{2}$.

3
a) $-\frac{1}{2}e^{-x^2} + C$
b) $\frac{1}{3}\sin^3(\theta) + C$
c) $\frac{1}{4}\ln\left|x^4 - 5\right| + C$

4
a) $\frac{2}{3}\sqrt{3} - 1$.
b) $\frac{32}{3} - 4\sqrt{3}$
c) $e - \sqrt{e}$
d) $\frac{a^3}{3}(2\sqrt{2} - 1)$

5
a) $\dfrac{x^3}{3} + x + \arctan(x) + C$
b) $2x + \dfrac{3^x}{\ln(3)} + C$
c) $\theta + \tan\theta + C$

6
a) $1 + \dfrac{\pi}{4}$
b) $\dfrac{1}{2}$
c) $-\dfrac{7}{2}$
d) 3

7
a) $-\dfrac{2}{3}\left(1 + \dfrac{1}{x}\right)^{3/2} + C$
b) $2\sin\sqrt{t} + C$

8
a) $\ln(e + 1)$
b) $2\arctan 2 - \dfrac{\pi}{2}$
c) $\dfrac{1}{6}$
d) $\dfrac{4}{3}\left(7 - \ln\dfrac{9}{2}\right)$